

Electrostatic solutions in Kaluza-Klein theory: geometry and stability

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Abstract

We investigate the family of electrostatic spherically symmetric solutions of the five-dimensional Kaluza-Klein theory. Both charged and neutral cases are considered. The analysis of the solutions, through their geometrical properties, reveals the existence of black holes, wormholes and naked singularities. A new class of regular solutions is identified. A monopole perturbation study of all these solutions is carried out, enabling us to prove analytically the stability of large classes of solutions. In particular, the black hole solutions are stable, while for the regular solutions the stability analysis leads to an eigenvalue problem.

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1 Introduction

Five-dimensional Kaluza–Klein theory [1, 2], or sourceless general relativity in 4+1 space-time dimensions (the extra space dimension being compactified), was historically one of the first unified field theories. While this simplest higher-dimensional field certainly cannot be considered as realistic, it nevertheless deserves to be investigated as the prototype of other multidimensional theories. The static, spherically symmetric solutions of Kaluza–Klein theory have been obtained independently by several authors [3, 4, 5]. These solutions include regular black holes, which are generalisations of Schwarzschild black holes with electric and scalar charges. A systematic investigation of these black hole solutions was carried out by Gibbons and Wiltshire [6]. A class of regular, horizonless charged solutions with wormhole spatial topology was also identified by Chodos and Detweiler in [5]. Apart from these black hole and wormhole solutions, all other solutions apparently possess naked singularities.

The aim of the present work is to analyze more fully the geometrical properties of the static spherically symmetric solutions of Kaluza–Klein theory, as well as to investigate their stability. As far as we know, the first systematic examination of the stability of these solutions is that of Tomimatsu [7], which was restricted to electrically neutral solutions. Tomimatsu concluded that the only stable neutral solution was the 5-dimensional embedding of the Schwarzschild black hole. The stability of a class of wormhole solutions was analysed in [8]; the conclusion was that these were generically stable. This investigation was generalized in an unpublished work [9] to encompass all the static spherically symmetric solutions. However, our subsequent analysis of the stability of scalar-tensor black holes [10] led us to uncover a flaw in the arguments of [8, 9], which motivated us to launch a systematic reinvestigation of the Kaluza–Klein problem.

In the second section of this paper, we recall the construction of the static, spherically symmetric (in the three “external” space dimensions) solutions to Kaluza–Klein theory. The solutions depend generally on three parameters x , a and b , related to the mass, scalar charge and electrical charge. The analysis of the geometrical invariants and the geodesics, carried out in section 3, allows the identification of the different solutions as black holes, wormholes and naked singularities in the five dimensional space-time. Some of these naked singularities turn out to be at an infinite geodesic distance, so that the corresponding solutions are regular. The equations for small time-dependent monopole perturbations of these solutions are then set up in a gauge-independent fashion in Sect. 4, and decoupled in a special gauge. The transformation of these perturbations under gauge transformations is also briefly discussed.

Using this framework, the question of the stability of static solutions to Kaluza–Klein theory is reduced to an eigenvalue problem. An analytical investigation of this problem is carried out in Sect. 5. We are able to prove that two classes of solutions are stable. The first stability class includes (contrary to Tomimatsu’s claim) all the electrically neutral solutions, while the second stability class includes, among others, all black hole and extreme black hole solutions. We then discuss two special cases in which we are able to prove instability (the second one containing a stable subcase). In the remaining cases, where we have no analytical information about the spectrum of eigenmodes, one should

resort to numerical computations to ascertain whether the corresponding solutions are stable under monopole perturbations.

2 Dimensional reduction and electrostatic solutions

The field equations of Kaluza–Klein theory derive from the 5–dimensional Einstein–Hilbert action

$$S = -\frac{1}{16\pi G_5} \int d^5x \sqrt{|g_5|} R_5, \quad (2.1)$$

with the additional assumption that $\partial/\partial x^5$ is a Killing vector with closed orbits. This last assumption allows the 5–dimensional metric to be decomposed as

$$ds_5^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu - e^{2\psi} (dx^5 + 2A_\mu dx^\mu)^2. \quad (2.2)$$

The 5–dimensional Einstein equations then reduce to the 4–dimensional system

$$\bar{R}_\mu{}^\nu = -2e^{2\psi} F_{\mu\rho} F^{\nu\rho} + e^{-\psi} \bar{D}_\mu \bar{D}^\nu e^\psi, \quad (2.3)$$

$$\bar{D}_\nu (e^{3\psi} F^{\mu\nu}) = 0, \quad (2.4)$$

$$\square e^\psi = -e^{3\psi} F_{\mu\nu} F^{\mu\nu}, \quad (2.5)$$

with $F_{\mu\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu}$.

The first of these equations exhibits a non–minimal coupling of the scalar field e^ψ to 4–dimensional gravity. A minimal coupling is recovered by making the conformal transformation $\bar{g}_{\mu\nu} = e^{-\psi} g_{\mu\nu}^E$ to the Einstein frame, leading to

$$ds_5^2 = e^{-2\varphi/\sqrt{3}} g_{\mu\nu}^E dx^\mu dx^\nu - e^{4\varphi/\sqrt{3}} (dx^5 + 2A_\mu dx^\mu)^2, \quad (2.6)$$

with the dilaton field $\varphi = \sqrt{3}\psi/2$. This Einstein frame, frequently used in the literature [6], is not defined when $g_{55} = -e^{2\psi}$ is not negative definite, which will be the case for a large class of the solutions discussed in this paper. For this reason, we will use only the string frame defined in (2.2). Let us also note that for $A_\mu = 0$ the equations (2.3)–(2.5) reduce to the Jordan–frame equations of Brans–Dicke theory for $\omega = 0$ [11].

The spherically symmetric, electrostatic solutions of Kaluza–Klein theory have been previously obtained by several authors [3, 4, 5]. To be self–contained, we shall rederive them here along the lines of [4], using the Maison approach to the dimensional reduction of the higher–dimensional Einstein equations [12], which we first briefly summarize. The metric of $(n + p)$ –dimensional spacetime with p commuting Killing vectors may be parametrized by

$$ds^2 = \lambda_{ab} (dx^a + \mathcal{A}_i^a dx^i) (dx^b + \mathcal{A}_j^b dx^j) + \tau^{-1} h_{ij} dx^i dx^j, \quad (2.7)$$

where $i = 1, \dots, n$, $a = n + 1, \dots, n + p$, $\tau = |\det(\lambda)|$, and the various fields depend only on the coordinates x^i . In our case $n = 3$, $p = 2$, x^4 is the time coordinate and x^5 is the Kaluza–Klein angular coordinate. Using the $(n + p)$ –dimensional Einstein equations,

the magnetic-like vector potentials \mathcal{A}_i^a may be dualized to the scalar twist potentials ω_a according to

$$\omega_{a,i} \equiv |h|^{-1/2} \tau \lambda_{ab} h_{ij} \epsilon^{jkl} \mathcal{A}_{j,k}^b. \quad (2.8)$$

The remaining Einstein equations may then be written as the n -dimensional Einstein- σ -model system

$$(\chi^{-1} \chi^i)_{;i} = 0, \quad (2.9)$$

$$R_{ij} = \frac{1}{4} \text{Tr}(\chi^{-1} \chi_{;i} \chi^{-1} \chi_{;j}), \quad (2.10)$$

where the n -metric is h_{ij} , and χ is the (anti-)unimodular symmetric $(p+1) \times (p+1)$ matrix-valued field

$$\chi = \begin{pmatrix} \lambda_{ab} + \tau^{-1} \omega_a \omega_b & \tau^{-1} \omega_a \\ \tau^{-1} \omega_b & \tau^{-1} \end{pmatrix}. \quad (2.11)$$

Solutions of Eq. (2.9) depending on a single potential $\sigma(x^i)$ are geodesics

$$\chi = \eta e^{A\sigma} \quad (2.12)$$

of the target space $\text{SL}(p+1, \mathbb{R})/\text{SO}(p+1)$, η and A being real constant matrices (with $|\det \eta| = 1$, $\text{Tr}(A) = 0$, $\eta^T = \eta$, $A^T \eta = \eta A$) and σ a harmonic function,

$$\nabla^2 \sigma = 0. \quad (2.13)$$

Now we specialize to $n = 3$, $p = 2$, and restrict ourselves to electrostatic solutions, $\omega_a = 0$. If $\sigma(\infty) = 0$, the metric (2.7) is asymptotically Minkowskian provided

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.14)$$

Then the 3×3 matrix A is block-diagonal and may be parametrized by

$$A = \begin{pmatrix} N & 0 \\ 0 & -x \end{pmatrix}, \quad N = \begin{pmatrix} x-a & b \\ -b & a \end{pmatrix}. \quad (2.15)$$

For a spherically symmetric solution with the potential $\sigma(r)$ normalized by $\lim_{(r \rightarrow \infty)} r \sigma(r) = -1$, the parameters x , a and b are related to the physical observables M (ADM mass), Σ (scalar charge) and Q (electric charge) by [13, 14]

$$x = 2(M - \Sigma/\sqrt{3}), \quad a = -4\Sigma/\sqrt{3}, \quad b = 2Q. \quad (2.16)$$

We may choose for the spherically symmetric reduced spatial metric the parametrization

$$-h_{ij} dx^i dx^j = dr^2 + H(r) d\Omega^2. \quad (2.17)$$

The harmonic function $\sigma(r)$ then solves

$$\sigma_{,r} = H^{-1}. \quad (2.18)$$

Inserting the ansatz (2.17) in the reduced Einstein equation (2.10), we obtain the system,

$$\begin{aligned} 1 - \frac{1}{2}H_{,rr} &= 0, \\ -H^{-1}H_{,rr} + \frac{1}{2}H^{-2}H_{,r}^2 &= 2\nu^2 H^{-2}, \end{aligned} \quad (2.19)$$

with

$$\nu^2 = \frac{1}{4}(x^2 - y), \quad (2.20)$$

where

$$y \equiv \det N = b^2 + ax - a^2. \quad (2.21)$$

This system is solved by

$$H(r) = r^2 - \nu^2. \quad (2.22)$$

The form of the function $\sigma(r)$ obtained by integrating Eq. (2.18) depends on the sign of the constant ν^2 :

a) $y < x^2$ ($\nu^2 > 0$).

In this case,

$$\sigma = \frac{1}{2\nu} \ln \left(\frac{r - \nu}{r + \nu} \right) \quad (2.23)$$

diverges for $r = \nu$. From the form of the function $\tau = e^{x\sigma}$, we see that $r = \nu$ is for $x < 2\nu$ ($x < 0$ or $x \geq 0, y < 0$) a point singularity of the 5-dimensional metric, and for $x \geq 2\nu$ ($x \geq 0, y \geq 0$) a Killing horizon.

b) $y = x^2$ ($\nu^2 = 0$).

In this “extreme” case the function

$$\sigma = -\frac{1}{r} \quad (2.24)$$

diverges for $r = 0$, which is a point singularity if $x < 0$.

c) $y > x^2$ ($\nu^2 < 0$).

In this case, which corresponds to the “Class III” of [5], the function

$$\sigma = -\frac{1}{\mu} \left(\frac{\pi}{2} - \arctan \left(\frac{r}{\mu} \right) \right), \quad (2.25)$$

where

$$\mu^2 = -\nu^2 = \frac{1}{4}(y - x^2),$$

varies within the finite range $I_\sigma =] -\pi/\mu, 0[$ when r varies from $r \rightarrow -\infty$ to $r \rightarrow +\infty$. We therefore anticipate that the resulting 5-dimensional metric will be exempt from singularities.

The full 5-dimensional metric (2.7) may be written down once the matrix $\lambda(r)$, given by (2.12) and (2.11), is known. The exponential in (2.12) is easily computed with the aid of the Lagrange interpolation formula [15]

$$f(A) = \sum_i f(z_i) \prod_{j \neq i} \frac{A - z_j}{z_i - z_j}. \quad (2.26)$$

Its form depends on the sign of

$$q^2 \equiv x^2/4 - y, \quad (2.27)$$

which determines whether the eigenvalues z_i of the matrix A are real, degenerate or complex. We obtain in these three cases:

$$\begin{aligned} \lambda &= \frac{1}{q} e^{x\sigma/2} \begin{pmatrix} \left(\frac{x}{2} - a\right) \sinh q\sigma + q \cosh q\sigma & b \sinh q\sigma \\ b \sinh q\sigma & \left(\frac{x}{2} - a\right) \sinh q\sigma - q \cosh q\sigma \end{pmatrix} \quad \text{for } y < \frac{x^2}{4}; \\ \lambda &= e^{x\sigma/2} \begin{pmatrix} \left(\frac{x}{2} - a\right) \sigma + 1 & b\sigma \\ b\sigma & \left(\frac{x}{2} - a\right) \sigma - 1 \end{pmatrix} \quad \text{for } y = \frac{x^2}{4}; \\ \lambda &= \frac{1}{p} e^{x\sigma/2} \begin{pmatrix} \left(\frac{x}{2} - a\right) \sin p\sigma + p \cos p\sigma & b \sin p\sigma \\ b \sin p\sigma & \left(\frac{x}{2} - a\right) \sin p\sigma - p \cos p\sigma \end{pmatrix} \quad \text{for } y > \frac{x^2}{4}, \end{aligned} \quad (2.28)$$

where $p^2 \equiv y - x^2/4$. In all cases it can be checked that $\tau = e^{x\sigma}$. We will also use, besides the 5-dimensional metric (2.7), the reduced 4-dimensional fields $\bar{g}_{\mu\nu}$, A_μ and $e^{2\psi}$ obtained from (2.2), which we shall write in the form

$$d\bar{s}^2 = e^{2\gamma} dt^2 - e^{2\alpha} d\rho^2 - e^{2\beta} d\Omega^2, \quad A_\mu dx^\mu = V dt, \quad (2.29)$$

with

$$e^{2\alpha} = -e^{-x\sigma} h_{\rho\rho}, \quad e^{2\beta} = H e^{-x\sigma}, \quad e^{2\gamma} = e^{x\sigma - 2\psi}, \quad e^{2\psi} = -\lambda_{55}, \quad V = \lambda_{45}/2\lambda_{55}. \quad (2.30)$$

If ρ in (2.29) is identical with the radial coordinate r defined in (2.17), $e^{2\alpha} = e^{-x\sigma}$.

3 Black holes and regular solutions

3.1 Case $y < x^2$.

As observed in the preceding section, the surface $r = \nu$ ($\sigma \rightarrow -\infty$) where $\tau = |\det(\lambda)|$ vanishes is a Killing horizon of the 5-dimensional metric, as well as of the reduced 4-dimensional metric (2.29), for $x \geq 0$, $y \geq 0$ [6]. However the 5-dimensional metric (2.7) and the reduced 4-dimensional fields (2.30) are analytical only for $y = 0$.

On the Killing horizon $r = \nu$, the matrix λ reduces for $y = 0$, $x > 0$ to

$$\lambda_H = \frac{1}{x} \begin{pmatrix} a & -b \\ -b & a - x \end{pmatrix}, \quad (3.1)$$

with eigenvalues 0 and $(2a - x)/x$. If $a \leq 0$, then $2a \leq 0 < x$, so that the non-zero eigenvalue $(2a - x)/x$ is negative and the non-null Killing directions are spacelike,

corresponding to an event horizon. The black-hole solutions of this class [4], which includes the 5-dimensional Schwarzschild metric for $a = b = 0$, have been extensively studied by Gibbons and Wiltshire [6]. On the other hand, if $a > 0$, then (from $y = 0$) $x = a - b^2/a < a < 2a$, so that the non-null Killing directions are timelike, and the Killing horizon is actually a conical singularity. The neutral solution $b = 0$ ($x = a$)

$$ds^2 = dt^2 - \frac{r - \nu}{r + \nu} (dx^5)^2 - \frac{r + \nu}{r - \nu} dr^2 - (r + \nu)^2 d\Omega^2 \quad (3.2)$$

is the direct product of the time axis by the Euclidean Schwarzschild metric, and is regular if the period of x^5 is $8\pi\nu$ [5]. The general $b \neq 0$ solution behaves near $r = \nu$ as the product of the (2+1)-dimensional metric generated by a spinning point particle [16] by the 2-sphere S_2 .

In the case $y \neq 0$, it is generally assumed that $r = \nu$ is a curvature singularity [4][6]. This assumption is based on an extrapolation from the neutral case $b = 0$. However the extrapolation goes through only if $y < x^2/4$, in which case the matrix λ has real eigenvalues and can always be diagonalized, so that the 5-dimensional curvature invariants of a charged solution are identical with those of some neutral solution. On the other hand, if $y \geq x^2/4$, then the 5-dimensional metric cannot be diagonalized, and we must compute the metric invariants to ascertain whether they diverge on the Killing horizon.

The components of the 5-dimensional curvature tensor may be computed from the formulas given in [9]. Using the parametrization (2.15), (2.17) and Eq. (2.18), we obtain

$$\begin{aligned} R_{abcd} &= \frac{1}{4} \tau H^{-2} [(\lambda N)_{ac}(\lambda N)_{bd} - (\lambda N)_{ad}(\lambda N)_{bc}] , \\ R_{arrb} &= \frac{1}{4} H^{-2} [\lambda (N^2 + (x - 4r)N)]_{ab} , \\ R_{a\theta\theta b} &= \frac{1}{4} H^{-1} (2r - x)(\lambda N)_{ab} , \\ R_{r\theta r\theta} &= \frac{1}{4} \tau^{-1} H^{-1} (y - x^2 + 2xr) , \\ R_{\theta\varphi\theta\varphi} &= \frac{1}{4} \tau^{-1} \sin^2 \theta (2x^2 - y - 4xr) . \end{aligned} \quad (3.3)$$

The computation of the 5-dimensional Kretschmann invariant $K_5 = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$ is easily carried out by tracing powers of the matrix N , for instance

$$R^{ab}_{cd} R^{cd}_{ab} = \frac{1}{8} \tau^2 H^{-4} ([\text{Tr}(N^2)]^2 - \text{Tr}(N^4)) , \quad (3.4)$$

leading to the compact result

$$K_5 = \frac{3}{2} \tau^2 H^{-4} [y^2 + 4\rho xy + 8\rho^2(x^2 - y)] , \quad (3.5)$$

with $\rho \equiv r - x/2$.

From (3.5) we obtain the behaviour of the Kretschmann scalar near the Killing horizon ($r \simeq \nu$),

$$K_5 \simeq \frac{3(x - 2\nu)^2(2x^2 - 3y)}{512\nu^8} \left(\frac{r - \nu}{2\nu} \right)^{(x-4\nu)/\nu} . \quad (3.6)$$

For $\nu > x/4$ ($y < 3x^2/4$) this diverges on the Killing horizon, except in the special cases $\nu = x/2$, which corresponds to the $y = 0$ black holes, or $y = 2x^2/3$ ⁵. On the other hand, for $\nu \leq x/4$ ($3x^2/4 \leq y < x^2$) the 5-dimensional Kretschmann scalar is finite or vanishes on the Killing horizon. The form of the Kretschmann scalar K_4 for the reduced 4-dimensional metric (2.29) is more involved as it depends on that of the Kaluza–Klein scalar field λ_{55} . We have found that on the Killing horizon $r = \nu$, it likewise diverges for $y < 3x^2/4$ (except for $y = 0$), and is finite or vanishes for $3x^2/4 \leq y < x^2$.

Finite curvature invariants often, but not always, signal regularity of the spacetime geometry. A recent counter-example is that of “cold black hole” solutions [10, 17], which have everywhere finite curvature invariants but, because of a lack of analyticity which prevents a Kruskal-like extension through the Killing horizon, are generically singular, except for a discrete set of solutions. In the present case we will see that the $3x^2/4 \leq y < x^2$ solutions are indeed regular, although non-analytical on the horizon. They need not be extended through the Killing horizon because this is at infinite geodesic distance. In other words, they are already geodesically complete.

The energy integral for geodesic motion in the spherically symmetric 5-dimensional metric (2.7)–(2.17) may be written as

$$\dot{r}^2 + \frac{l^2 \tau^2}{r^2 - \nu^2} + V(r) = 0, \quad V \equiv \tau(\eta - \Pi^T \lambda^{-1} \Pi), \quad (3.7)$$

where Π_4 , Π_5 and l are constants of the motion proportional to the energy, electric charge and angular momentum of the test particle, $\dot{r} \equiv dr/d\rho$ with ρ an affine parameter, and the (arbitrarily scaled) integration constant η is positive, zero or negative for timelike, null or spacelike geodesics. Assuming $x^2/4 < y < x^2$, $x > 0$ (the Killing horizon can be shown to be at finite geodesic distance for $y \leq x^2/4$, $x > 0$), we obtain from (2.28) the effective potential

$$V = e^{x\sigma} (\eta - \alpha e^{-x\sigma/2} \sin(p\sigma + \beta)) \quad (3.8)$$

where the constants α and β depend on Π_4 and Π_5 .

For $l = 0$, $e^{-x\sigma/2}$ increases when when the horizon $r = \nu$ ($\sigma \rightarrow -\infty$) is approached so that, whatever the values of the constants of the motion, with $\alpha \neq 0$, there is a σ_1 such that

$$e^{-x\sigma_1/2} \sin(p\sigma_1 + \beta) = \eta/\alpha, \quad (3.9)$$

corresponding to a turning point of the geodesic (reflection on a potential barrier). The only geodesics which can reach the horizon are those with $\alpha = 0$ ($\Pi_4 = \Pi_5 = 0$) and $\eta < 0$, leading to

$$\rho \sim (r - \nu)^{(4\nu - x)/4\nu}. \quad (3.10)$$

It then follows that:

a) If $x^2/4 < y < 3x^2/4$ ($x < 4\nu$), those geodesics terminate on the horizon (on which, as shown previously, the curvature invariant K_5 diverges). The 5-dimensional spacetime is geometrically singular, although this singularity is harmless to physical test particles

⁵However the fact that in this case the Kretschmann scalar vanishes on the horizon does not seem relevant, as the 5-dimensional metric is not analytical and cannot be extended through the horizon.

($\eta \geq 0$), which will be reflected away before hitting the singularity (the same can be shown to be true for $y = x^2/4$ if $x < 2a$).

b) If $3x^2/4 \leq y < x^2$ ($x \geq 4\nu$), the horizon (on which $K_{(5)}$ is finite or vanishes) is at infinite geodesic distance, so that the 5-dimensional spacetime is geodesically complete. While these 5-dimensional spacetimes are geometrically regular, they admit closed time-like curves. From (2.28), λ_{55} changes sign periodically, so that all the circles $r = \theta = \varphi = t = \text{const.}$ are timelike in the domains where $\lambda_{55} > 0$. This is linked with the fact that the reduced 4-dimensional metric $g_{\mu\nu}$ in (2.2) is singular on the spheres $\lambda_{55} = 0$, which separate 4-Minkowskian regions ($\lambda_{55} < 0$) from 4-Euclidean regions ($\lambda_{55} > 0$). We take the view that these 4-dimensional singularities are artefacts due to the breakdown of 5-to-4 dimensional reduction, i.e. the choice of a bad coordinate system (2.2) for the 5-dimensional geometry, which is perfectly regular in the parametrization (2.7).

3.2 Case $y = x^2$.

In this case $\nu = 0$ and $\sigma = -1/r$. We again assume $x \geq 0$, as the 5-dimensional geometry is obviously singular at $r = 0$ if $x < 0$. In the case $x = 0$, then $y = 0$ and $b = \pm a$, leading to the 5-dimensional “extreme black hole” metric with flat spatial sections [4, 13]

$$ds^2 = \left(1 + \frac{a}{r}\right) dt^2 \mp \frac{2a}{r} dt dx^5 - \left(1 - \frac{a}{r}\right) (dx^5)^2 - dr^2 - r^2 d\Omega^2. \quad (3.11)$$

While from (3.5) the Kretschmann scalar vanishes in this case, the study of geodesic motion shows that this geometry is actually singular. The effective potential

$$V = \eta + \Pi_5^2 - \Pi_4^2 + (\Pi_4 \pm \Pi_5)^2 \frac{a}{r} \quad (3.12)$$

is unbounded from below for $a < 0$, in which case all the $l = 0$ geodesics terminate at the singularity $r = 0$. For $a > 0$, only geodesics with $l = \Pi_4 \pm \Pi_5 = 0$, $\eta < 0$ terminate at the singularity, which is thus harmless to physical test particles.

The general case $x > 0$ is quite similar to the case b) above, the 5-dimensional spacetime being geodesically complete, with an infinite number of changes of sign of λ_{55} .

3.3 Case $y > x^2$.

Inspection of the 5-dimensional metric (2.7), (2.28) with σ given by (2.25) shows that for all values of x it is regular [5] for $r \in]-\infty, +\infty[$, so that the 5-dimensional geometry is of the Lorentzian wormhole type. However, as pointed out in [18] (in the special case $x = 0$ of the symmetrical wormhole), these wormholes are non-traversable, in the sense that physical (non-tachyonic) test particles cannot go from one asymptotically flat region ($r \rightarrow +\infty$) to the other ($r \rightarrow -\infty$). This would be possible only if the effective potential $V(r)$ in (3.7) was negative over the whole range of r . Here the potential (3.8) with $\eta \geq 0$ is necessarily positive over part of this range, as the range $] -p\pi/\mu, 0[$ of $p\sigma$ is at least 2π ($p^2 \geq 4\mu^2$), and a physical test particle coming from $r \rightarrow +\infty$ is always reflected back to $r \rightarrow +\infty$, just as in the case $x^2/4 < y \leq x^2$.

Likewise, 5-dimensional light cones also gradually tumble over when r decreases from $+\infty$ to $-\infty$, leading to the existence of closed timelike curves. The 5-dimensional metric is asymptotically Minkowskian at both points at spatial infinity if $p = 2n\mu$ ($y/x^2 = (n^2 - 1/4)/(n^2 - 1)$) with n integer, in which case light cones tumble over by $n\pi$ between the two points at infinity, and the solution is a metrical kink of winding number n [19] (other axisymmetric kink solutions of Kaluza–Klein theory are discussed in [20]).

4 Small perturbations

In this section we set up the equations for small time-dependent spherically symmetric perturbations of the electrostatic solutions. As shown in [8], we may choose for the five-dimensional metric a parametrization similar to that of (2.2), (2.29) with metric functions now depending on time,

$$ds_5^2 = e^{2\gamma(\rho,t)} dt^2 - e^{2\alpha(\rho,t)} d\rho^2 - e^{2\beta(\rho,t)} d\Omega^2 - e^{2\psi(\rho,t)} (dx^5 + 2V(\rho, t) dt)^2. \quad (4.1)$$

The Kaluza–Klein Gauss law (Eq. (2.4)) may be integrated to

$$F^{41} = Q e^{-\alpha-2\beta-\gamma-3\psi}, \quad (4.2)$$

with Q the conserved electric charge. The remaining time-dependent spherically symmetric equations (2.3) and (2.5) then reduce to the system

$$e^{-2\beta+\alpha-\gamma-\psi} (e^{2\beta-\alpha+\gamma+\psi} \psi')' - e^{2(\alpha-\gamma)} \ddot{\psi} = -2Q^2 e^{2(\alpha-2\beta-2\psi)}, \quad (4.3)$$

$$e^{-2\beta+\alpha-\gamma-\psi} (e^{2\beta-\alpha+\gamma+\psi} \beta')' - e^{2(\alpha-\gamma)} \ddot{\beta} = e^{2(\alpha-\beta)}, \quad (4.4)$$

$$e^{-2\beta+\alpha-\gamma-\psi} (e^{2\beta-\alpha+\gamma+\psi} \gamma')' - e^{2(\alpha-\gamma)} (\ddot{\alpha} + 2\ddot{\beta} + \ddot{\psi}) = 2Q^2 e^{2(\alpha-2\beta-2\psi)}, \quad (4.5)$$

$$-2\beta'' - \psi'' + \beta'(2\alpha' - 3\beta') + \psi'(\alpha' - 2\beta' - \psi') + e^{2(\alpha-\beta)} = Q^2 e^{2(\alpha-2\beta-2\psi)}, \quad (4.6)$$

$$-2\dot{\beta}' - \dot{\psi}' + (2\beta' + \psi')\dot{\alpha} + 2(-\beta' + \gamma')\dot{\beta} + (\gamma' - \psi')\dot{\psi} = 0 \quad (4.7)$$

with $' = \partial/\partial\rho$, $\dot{} = \partial/\partial t$. These equations are not all independent. The first three dynamical equations correspond respectively to Eq. (2.5) and to the \bar{R}_2^2 and \bar{R}_4^4 components of Eq. (2.3), while the last two constraint equations correspond to $(\bar{R}_4^4 - \bar{R}_i^i)/2$ and \bar{R}_{41} respectively.

Now we linearize the metric fields in (4.1) around the static background fields in Eqs. (2.29),

$$\psi(\rho, t) = \psi(\rho) + \delta\psi(\rho, t), \text{ etc}, \quad (4.8)$$

where $\delta\psi, \delta\alpha, \delta\beta, \delta\gamma$ are small perturbations. The linearization of equations (4.3)–(4.7) leads to the differential system for the perturbations, which has been simplified by using the static equations of motion, and choosing a “harmonic” background coordinate system in which the background fields $\psi, \alpha, \beta, \gamma$ are related by

$$\alpha - 2\beta - \gamma - \psi = 0 \quad (4.9)$$

(which amounts to choosing the radial coordinate $\rho = \sigma$, with $h_{\rho\rho} = H^2$):

$$\begin{aligned} \delta\psi'' - \psi'\delta\alpha' + 2\psi'\delta\beta' + \psi'\delta\gamma' + \psi'\delta\psi' \\ - e^{2(2\beta+\psi)}\delta\ddot{\psi} - 2\psi''(\delta\alpha - 2\delta\beta - 2\delta\psi) = 0, \end{aligned} \quad (4.10)$$

$$\begin{aligned} \delta\beta'' - \beta'\delta\alpha' + 2\beta'\delta\beta' + \beta'\delta\gamma' + \beta'\delta\psi' \\ - e^{2(2\beta+\psi)}\delta\ddot{\beta} - 2\beta''(\delta\alpha - \delta\beta) = 0, \end{aligned} \quad (4.11)$$

$$\begin{aligned} \delta\gamma'' + \gamma'(-\delta\alpha' + 2\delta\beta' + \delta\gamma' + \delta\psi') - e^{2(2\beta+\psi)}(\delta\ddot{\alpha} \\ + 2\delta\ddot{\beta} + \delta\ddot{\psi}) + 2\psi''(\delta\alpha - 2\delta\beta - 2\delta\psi) = 0, \end{aligned} \quad (4.12)$$

$$\begin{aligned} 2\delta\beta'' + \delta\psi'' - (2\beta' + \psi')\delta\alpha' + 2(\beta' - \gamma')\delta\beta' + (\psi' - \gamma')\delta\psi' \\ - (2\beta'' + \psi'')\delta\alpha + 2(\beta'' + \psi'')\delta\beta + 2\psi''\delta\psi = 0, \end{aligned} \quad (4.13)$$

$$2\delta\dot{\beta}' + \delta\dot{\psi}' = (2\beta' + \psi')\delta\dot{\alpha} + 2(-\beta' + \gamma')\delta\dot{\beta} + (\gamma' - \psi')\delta\dot{\psi}. \quad (4.14)$$

The last two constraint equations are not independent, as the time derivative of (4.13) may be seen (using $\gamma'' + \psi'' = 0$, which follows from (2.30)) to be identical with the space derivative of (4.14).

The perturbations $\delta\psi$, $\delta\alpha$, etc. are defined only up to a change of coordinates preserving the form of the metric (4.1), so that we still have the “gauge freedom” to choose coordinates for the perturbed spacetime by imposing a supplementary relation between these perturbations. The separation of the linearized equations (4.10)–(4.14) is simpler in the gauge where the perturbations $\delta_1\psi$, $\delta_1\alpha$, etc. are constrained by the gauge condition⁶

$$2\delta_1\beta + \delta_1\psi = 0. \quad (4.15)$$

The integration of the constraint equation (4.14) then leads to the relation

$$\delta_1\alpha = \frac{\psi' - \beta'}{\psi' + 2\beta'} \delta_1\psi. \quad (4.16)$$

With the help of the unperturbed field equations and Eq. (4.15), Eqs. (4.10), (4.11) combined together according to: $\beta' \times (4.10) - \psi' \times (4.11)$, lead to the wave equation for $\delta_1\psi$

$$\delta_1\psi'' - e^{2(2\beta+\psi)}\delta_1\ddot{\psi} - 6\frac{F'}{F^2}\delta_1\psi = 0, \quad (4.17)$$

with

$$F \equiv 2/\psi' + 1/\beta'. \quad (4.18)$$

From a solution $\delta_1\psi(\sigma, t)$ to this master equation, the other perturbations in the gauge (4.15) may be obtained by using Eqs. (4.15), (4.16), and the equation

$$\delta_1\gamma' = 2\frac{\psi' - \beta'}{\psi' + 2\beta'} \delta_1\psi' - \delta_1\alpha', \quad (4.19)$$

obtained by adding Eqs. (4.10) and (4.11) according to $(4.10) + 2 \times (4.11)$, and using, first Eq. (4.15) to simplify the obtained equation, then Eq. (4.16) to cancel ψ'' , β'' .

⁶In the Einstein frame (2.6) this gauge condition reads simply $\delta_1\beta_E = 0$.

One can also gauge transform these perturbations $\delta_1\psi$, $\delta_1\alpha$, etc. to obtain the perturbations $\delta\psi$, $\delta\alpha$, etc. in a generic gauge, by carrying out the coordinate transformation $(t_1, \rho_1) \rightarrow (t, \rho)$. For instance the scalar field ψ can be linearized around its static value in both coordinate systems,

$$\begin{aligned}\psi(t, \rho) &= \psi(\rho) + \delta\psi(t, \rho) + \dots \\ &= \psi(\rho_1) + \delta_1\psi(t_1, \rho_1) + \dots \\ &= \psi(\rho_1) + \psi'(\rho_1)\delta_1\rho(t_1, \rho_1) + \delta\psi(t, \rho) + \dots,\end{aligned}\tag{4.20}$$

where we have linearized the coordinate transformation according to $\rho \simeq \rho_1 + \delta_1\rho(t_1, \rho_1)$. To first order Eq. (4.20) leads to

$$\delta\psi = \delta_1\psi - \psi' \delta_1\rho.\tag{4.21}$$

Eliminating $\delta_1\rho$ between (4.21) and a similar equation for the scalar perturbation $\delta\beta$, and using the gauge condition (4.15), we obtain the relation

$$\beta'\delta\psi - \psi'\delta\beta = \frac{1}{2}(2\beta' + \psi')\delta_1\psi.\tag{4.22}$$

From this last relation it can be shown that our wave equation (4.17) is identical with the scalar wave equations (18) or (20) of [8] (see also [9]) derived in the gauge $\delta_2\beta = 0$, the relation between the Kaluza–Klein scalar perturbations,

$$\delta_1\psi = \frac{2\beta'}{2\beta' + \psi'}\delta_2\psi,\tag{4.23}$$

leading to the identification with the notations of [8],

$$\delta_1\psi \equiv -\Phi/2 \equiv -fR/2,\tag{4.24}$$

with $f \equiv e^{-2\psi} 2\beta'/(2\beta' + \psi')$.

5 Stability

We now address the question of stability of electrostatic solutions against radial perturbations. Given stationary perturbations of the form $\delta_1\psi(\sigma, t) = \delta_1\psi(\sigma) e^{i\Omega t}$, etc., we search for non-trivial real solutions to the equation obtained from (4.17) by assuming Ω imaginary, $\Omega = -ik$ ($k > 0$),

$$\delta_1\psi'' - (k^2 e^{2(2\beta+\psi)} + 6F'/F^2)\delta_1\psi = 0,\tag{5.1}$$

where F is defined in (4.18). The existence of such solutions, satisfying some physical boundary conditions, means that any initially small perturbation will grow exponentially in time, thus the background solution is unstable. Conversely, the background solution is stable if all the eigenvalues Ω are real.

The effective potential function $U(\sigma)$ —the coefficient of $\delta_1\psi$ in the second term of Eq. (5.1)—has double poles at the zeroes σ_i of F , i.e. the roots of $2\beta' + \psi' = 0$. Near such possible poles, the effective potential behaves as

$$U(\sigma) \simeq -\frac{12\beta_i'^2}{(2\beta_i'' + \psi_i'')} \frac{1}{(\sigma - \sigma_i)^2} = \frac{2}{(\sigma - \sigma_i)^2}, \quad (5.2)$$

with $\beta_i' = \beta'(\sigma_i)$, etc., from which follows the behaviour

$$\delta_1\psi \simeq \frac{C_1}{\sigma - \sigma_i} + C_2(\sigma - \sigma_i)^2 \quad (5.3)$$

(C_1 and C_2 integration constants). So $\delta_1\psi$ generically has poles at $\sigma = \sigma_i$. However such poles turn out to be spurious, being induced by the gauge fixing $2\delta_1\beta + \delta_1\psi = 0$. Indeed, it follows from Eq. (4.22) that perturbations $\delta\psi$ and $\delta\beta$ which are regular in a generic gauge lead, in the gauge (4.15), to a $\delta_1\psi$ with poles at σ_i . Conversely these poles can be removed by transforming to another gauge. Similarly, from Eq. (4.23) spurious poles also occur in the gauge $\delta_2\beta = 0$ at the zeroes of β' [10, 8, 9]⁷.

These spurious poles being discarded, divergences of the perturbation $\delta_1\psi$ can only occur at the two ends of the range I_σ of σ , and must be excluded by the choice of appropriate boundary conditions. We shall adopt here, as physically reasonable from the 5-dimensional general-relativistic point of view, the boundary conditions previously used in scalar-tensor theories [10], which state that the relative perturbations of the background fields must be finite at the boundary. Since our functions $\delta_1\alpha$, $\delta_1\beta$, $\delta_1\gamma$, $\delta_1\psi$ are relative perturbations of the background metric fields in (2.2), these boundary conditions read

$$|\delta_1\alpha| < \infty, \quad |\delta_1\beta| < \infty, \quad |\delta_1\gamma| < \infty, \quad |\delta_1\psi| < \infty, \quad \text{for } \sigma \in I_\sigma \quad (5.4)$$

(the last of these boundary conditions was termed “strong” boundary condition in Ref. [10]). The boundary conditions (5.4) lead after gauge transformation to similar boundary conditions in a generic gauge.

Eq. (5.1) behaves at spatial infinity ($r \rightarrow +\infty$, $\sigma \rightarrow 0_-$) as

$$\delta_1\psi'' - \frac{k^2}{\sigma^4} \delta_1\psi \simeq 0, \quad (5.5)$$

for all values of the parameters (x, y, a) , leading to the asymptotic behaviour

$$\delta_1\psi \simeq \sigma \left(c_1 e^{k/\sigma} + c_2 e^{-k/\sigma} \right). \quad (5.6)$$

Our boundary conditions are satisfied by choosing $c_2 = 0$.

A number of possible behaviours of Eq. (5.1) can occur at the lower end σ_{\min} of the range I_σ , depending on the values of the parameters (x, y, a) . In most cases [9], the corresponding general asymptotic solution will be a linear combination of a bounded and

⁷In Refs. [8] and [9], solutions for which such poles occurred were argued to be stable, on the basis that generic perturbations $\delta_2\psi$ were unbounded; however this argument is not gauge invariant.

an unbounded solution with coefficients c_3 and c_4 . Then the perturbation will remain bounded near σ_{\min} provided c_4 is fixed equal to zero. However, owing to the scale invariance of Eq. (5.1), the two conditions $c_2 = c_4 = 0$ can be satisfied simultaneously only for special values of the parameter k^2 , i.e. we have an eigenvalue problem. In the absence of the knowledge of the exact solution in the whole interval, it is impossible in general to solve analytically this eigenvalue problem, and thus to establish with certainty the stable or unstable character of the background solution. However there are two parameter ranges, corresponding to neutral solutions, and to a preferred set of charged solutions, for which stability can be proved analytically by straightforward arguments. We will examine these cases in the next two subsections, and discuss in a third subsection two special cases for which unstability can be proved analytically.

5.1 Stability of neutral solutions

In the parameter space (x, y, a) , the domain of neutral solutions is obtained from (2.21) by setting $b = 0$. Since a is real, this leads to the portion of the surface

$$a^2 - xa + y = 0, \quad (5.7)$$

for which $y \leq x^2/4$ ($q^2 > 0$, with $q = |x/2 - a|$). As recalled in the Introduction, neutral Kaluza–Klein theory corresponds to Brans–Dicke theory with $\omega = 0$. Therefore the proof, given in [10] that all static spherically symmetric solutions to Brans–Dicke theory are stable under radial perturbations carries over to the present case. For completeness, we sketch here this proof in the Kaluza–Klein setting.

In the neutral case, the static scalar equation (4.3) reduces to $\psi'' = 0$ in the coordinate system (4.9), so that the linearized equation (4.10) immediately decouples from the other equations in the harmonic gauge

$$\delta_0 \alpha - 2\delta_0 \beta - \delta_0 \gamma - \delta_0 \psi = 0. \quad (5.8)$$

The resulting wave equation for the perturbation $\delta_0 \psi$ is

$$\delta_0 \psi'' - k^2 e^{2(2\beta+\psi)} \delta_0 \psi = 0. \quad (5.9)$$

The ratio $\delta_0 \psi'' / \delta_0 \psi = k^2 e^{2(2\beta+\psi)}$ being regular and positive over $I_\sigma =]-\infty, 0[$, it is impossible to keep $\delta_0 \psi$ finite at both ends of I_σ . Our boundary conditions (5.4) cannot be satisfied, so that this case is stable.

Tomimatsu [7] investigated the stability of 2-static solutions with vanishing electric field (5.7). In the parameter space (x, y, a) , he considered, as shown in [9], only the surface portions: $a = x/2 - q$, $y \leq 0$, $x > 0$, and $a = x/2 + q$, $y > 0$, $x < 0$, with $y < x^2/4$. He concluded that all the solutions under consideration were unstable, except the case $y = a = 0$, $x \neq 0$ of the Schwarzschild solution which was found to be stable. The discrepancy with our conclusions can be explained, as discussed in [9], by the fact that Tomimatsu treated the Schrödinger eigenvalue problem for an auxiliary function whose direct physical meaning is not transparent, and used in effect boundary conditions less stringent than our conditions (5.4).

5.2 A set of stable charged solutions

Let us return to the wave equation (5.1) in the gauge (4.15). The ratio $\delta_1\psi''/\delta_1\psi$ is positive definite over I_σ ($\forall k^2 > 0$) provided the two conditions

$$e^{2\psi} \geq 0, \quad (5.10)$$

$$F' \geq 0, \quad (5.11)$$

are satisfied simultaneously. Then, if the range I_σ is infinite (which is the case for $y \leq x^2$), and if further F does not have a zero in this range,

$$F \neq 0, \quad (5.12)$$

it is impossible to satisfy our boundary conditions (5.4), and the corresponding static solution is stable (a zero of F would lead, as discussed above, to a pole of the perturbation $\delta_1\psi(\sigma)$, which would enable it to remain finite at both ends of I_σ with $\delta_1\psi''/\delta_1\psi > 0$ everywhere; as we show in the Appendix the condition (5.12) is always satisfied when the conditions (5.10) and (5.11) are fulfilled).

Eq. (2.28) shows that $e^{2\psi} = -\lambda_{55}$ changes sign periodically for $y > x^2/4$, so that condition (5.10) cannot be satisfied, while it is satisfied for $y \leq x^2/4$ if $x/2 - a \geq 0$, implying (on account of (2.21))

$$x/2 - a \geq |b|. \quad (5.13)$$

To investigate condition (5.11), we compute (for $y < x^2/4$)

$$\psi' = \frac{x}{4} + \frac{q}{2} \coth(q\sigma - \eta), \quad (5.14)$$

$$\beta' = -\frac{x}{2} - \nu \coth \nu\sigma, \quad (5.15)$$

where η is defined by $q = |b| \sinh \eta$, $x/2 - a = |b| \cosh \eta$ ($\eta > 0$). Eq. (4.18) then leads to

$$F' = \frac{q^2}{\psi'^2 \sinh^2(q\sigma - \eta)} - \frac{\nu^2}{\beta'^2 \sinh^2(\nu\sigma)}. \quad (5.16)$$

The determination of the range for which this function remains non-negative is carried out analytically in the Appendix. The conclusion is that our stability conditions (5.10)-(5.12) are satisfied if either

- a) $x = a - b^2/a$ (implying $y = 0$) with $-2|b| \leq a < 0$. This class of stable solutions, includes black holes ($x > 0$, $a \leq 0$) and extreme black holes ($x = 0$, $a < 0$);
- b) $2(a + |b|) \leq x < a - b^2/a$ (implying $y > 0$) with $-2|b| \leq a < -|b|$.

5.3 Charged solutions: two unstable cases, and a stable subcase

5.3.1 Case $y > 0$, $x = a = 0$

In this massless case ($M = \Sigma = 0$), the matrix λ in (2.28) reduces to

$$\lambda = \begin{pmatrix} \cos b\sigma & \sin b\sigma \\ \sin b\sigma & -\cos b\sigma \end{pmatrix}, \quad (5.17)$$

leading to a symmetrical Lorentzian wormhole spacetime, with

$$e^{2\alpha} = 1, \quad e^{2\beta} = r^2 + \mu^2, \quad e^{2\gamma} = e^{-2\psi} = (r^2 + \mu^2)/(r^2 - \mu^2) \quad (5.18)$$

($r = -\mu \cot(\mu\sigma)$, $\mu = b/2 = Q$). A deeper investigation of this solution is given in [18, 8]. Due to the symmetry of the solution under the change in the radial coordinate $r \rightarrow -r$ and—consequently—of the wave equation (5.1) (rewritten in terms of the coordinate r)

$$[(r^2 + \mu^2) \delta_1 \psi_{,r}]_{,r} - \left[k^2 (r^2 - \mu^2) + \frac{6\mu^2}{r^2} \right] \delta_1 \psi = 0, \quad (5.19)$$

the two end-points at spatial infinity ($r \rightarrow \pm\infty$) are merged, leading to only one divergence at $r = +\infty$, which can be cancelled by choosing $c_2 = 0$ in Eq. (5.6). At the lower end of the range $[0, +\infty[$ of r , the perturbation $\delta_1 \psi$ is unbounded (cf. Eq. (29) of [8] and Eq. (4.24))

$$\delta_1 \psi = \frac{c_3}{r^2} \left[1 + \frac{1}{6} \left(k^2 + \frac{2}{\mu^2} \right) r^2 + \dots \right] + c_4 r^3 \left[1 - \frac{1}{14} \left(k^2 - \frac{34}{3\mu^2} \right) r^2 + \dots \right]. \quad (5.20)$$

Actually the pole in (5.20) is gauge dependent and can be removed by transforming to the gauge $\delta_2 \beta = 0$, according to Eq. (4.23), which leads to

$$\begin{aligned} \delta_2 \psi &= \frac{r^2}{r^2 - \mu^2} \delta_1 \psi \\ &= -\frac{c_3}{\mu^2} \left[1 + \frac{1}{6} \left(k^2 + \frac{8}{\mu^2} \right) r^2 + \dots \right] - \frac{c_4 r^5}{\mu^2} \left[1 - \frac{1}{14} \left(k^2 - \frac{76}{3\mu^2} \right) r^2 + \dots \right]. \end{aligned} \quad (5.21)$$

This is well bounded for all values of the integration constants. However the constraint equation (4.14) now leads to

$$\begin{aligned} \delta_2 \alpha &= \frac{\delta_2 \psi' + (\psi' - \gamma') \delta_2 \psi}{2\beta' + \psi'} \\ &= \frac{c_3 \mu^2}{6r^2} \left(k^2 - \frac{4}{\mu^2} \right) + \dots, \end{aligned} \quad (5.22)$$

which is singular, unless $c_3(k) = 0$ (leading to an eigenvalue problem), or

$$k = 2/\mu. \quad (5.23)$$

We verify that—for this special eigenvalue—all the perturbations $\delta_2 \psi$, etc, are bounded $\forall r \in [0, +\infty[$. The massless charged symmetric wormhole solution is then unstable for it admits a mode of perturbation growing in time as $e^{2t/\mu}$.

5.3.2 Case $y = 0$, $a > x > 0$

The functions which appear in Eq. (5.1) are given by

$$e^\beta = \frac{x}{1 - e^{x\sigma}}, \quad (5.24)$$

$$e^{2\psi} = \frac{x-a}{x} + \frac{a}{x} e^{x\sigma}, \quad (5.25)$$

$$F = \frac{4x-3a}{ax} e^{-x\sigma} + \frac{3}{x}. \quad (5.26)$$

With the help of Eq. (A.2), we see that the third term in Eq. (5.1) vanishes identically if $4x - 3a = 0$, or behaves at the lower end of I_σ ($\sigma \rightarrow -\infty$) as:

$$\frac{6F'}{F^2} \simeq \frac{6ax^2}{3a-4x} e^{x\sigma}$$

if $4x - 3a \neq 0$. Since the second term in Eq. (5.1) behaves as a constant ($\sigma \rightarrow -\infty$):

$$e^{4\beta+2\psi} \simeq -x^3(a-x),$$

we can neglect the third term, and write Eq. (5.1) as

$$\delta_1 \psi'' + k^2 x^3 (a-x) \delta_1 \psi \simeq 0. \quad (5.27)$$

We obtain by integration

$$\delta_1 \psi \simeq c_3 \cos m\sigma + c_4 \sin m\sigma \quad (5.28)$$

(with $m = kx\sqrt{x(a-x)}$), which is bounded for all values of the integration constants. We must check that the other perturbations $\delta_1 \alpha$ and $\delta_1 \gamma$ are also bounded. Using the asymptotic behaviour

$$\frac{\psi' - \beta'}{\psi' + 2\beta'} \simeq \begin{cases} \frac{x-2a}{3a-4x} & \text{if } 3a-4x \neq 0 \\ \frac{1}{2} e^{-x\sigma} & \text{if } 3a-4x = 0. \end{cases} \quad (5.29)$$

we find from Eqs. (4.16) and (4.19) that for $a \neq 4x/3$ all the perturbations are finite at $\sigma \rightarrow -\infty$. It follows that the boundary conditions (5.4) are satisfied for all values of k if $c_2 = 0$ in (5.6), and the corresponding background solutions are unstable.

However if $a = 4x/3 = 2|b|$ ($\Sigma = -2\sqrt{3}M$, $|Q| = 2M$), $\delta_1 \alpha$ and $\delta_1 \gamma$ diverge, from (4.16), (4.19) and (5.29), as $e^{-x\sigma} \delta_1 \psi$. These divergences may not be removed by transforming to another gauge. For instance, in the gauge $\delta_2 \beta = 0$, $\delta_2 \psi \sim e^{x\sigma} \delta_1 \psi$ from (4.23), leading from the first equation (5.22) to a $\delta_2 \alpha$ diverging as $e^{-x\sigma}$ times a bounded function. A similar divergence results in a generic linear gauge ($\delta\beta + \lambda\delta\psi = 0$) from the analysis of Eqs. (4.14) and (4.22). So at least one of the perturbations is never bounded⁸, leading to the conclusion that this special subcase is stable.

⁸One can show that $\delta\alpha$ is bounded in the gauge $\delta\alpha - \delta\beta = 0$; however $\delta\gamma$ is not bounded in this gauge.

6 Conclusion

We have discussed the geometry of the 3-parameter family of static spherically symmetric solutions of 5-dimensional Kaluza–Klein theory. We have found that, besides the 2-parameter black hole class ($y = 0, x > 0, a \leq 0$) and the exceptional regular solution ($y = 0, a = x > 0$), this family contains a 3-parameter class of geodesically complete solutions ($y \geq 3x^2/4, x > 0$), which is larger than previously thought. These regular solutions are necessarily charged.

We have also investigated the stability of these solutions under radial perturbations. We have shown that all neutral solutions ($b = 0$) are stable. Among charged solutions, we have found two stability classes ($y = 0, a < 0, x \geq 3a/4$) and ($0 < y \leq x^2/4, -2|b| \leq a < -|b|$); the first stability class includes all the black hole and extreme black hole solutions. We have also been able to prove analytically the unstability of two lower-dimensional (in the 3-dimensional parameter space) classes of solutions. The remaining cases (which include the geodesically complete solutions) lead to eigenvalue problems. Pending a numerical investigation of these, we can only conjecture that the corresponding static solutions are unstable.

This work should be generalized in two directions. Electrostatic spherically symmetric solutions are only a subcase of the stationary spherically symmetric solutions which may be obtained from Eq. (2.12), and which are discussed in the extreme case in [13, 14]. The stability of these solutions could be investigated along the lines followed here. Also, the proof of stability under monopole perturbations is but a first step. The effect of higher multipole perturbations should also be ascertained before one can conclude to stability under all small perturbations, but this is a far more difficult task.

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Appendix

In this Appendix we determine (for $y < x^2/4$ and $x/2 - a \geq |b|$) the parameter domains for which the function

$$F'(\sigma) = \frac{q^2}{\psi'^2 \sinh^2(q\sigma - \eta)} - \frac{\nu^2}{\beta'^2 \sinh^2(\nu\sigma)} \quad (\text{A.1})$$

remains non-negative in the range $]-\infty, 0]$. There are 3 cases according to the value of y :

1) $y < 0$ ($q > \nu$) . For $\sigma \rightarrow -\infty$, $F' \simeq -\nu^2/\beta'^2 \sinh^2 \nu \sigma$, so that condition (5.11) is not satisfied.

2) $y = 0$ ($q = \nu = |x|/2$), i.e. $b^2 = a(a - x)$. Then

$$F' = \frac{3a - 4x}{a} e^{-x\sigma}. \quad (\text{A.2})$$

Remembering that $a \leq x/2$, we find that condition (5.11) is satisfied for $a < 0$, $x \geq 3a/4$, implying $b^2 \geq a^2/4$. Condition (5.12) is then also satisfied because

$$F(\sigma) \leq F(0) = 2/\psi'(0) = 4/a < 0. \quad (\text{A.3})$$

The corresponding domain of stability is

$$x = a - b^2/a, \quad -2|b| \leq a < 0. \quad (\text{A.4})$$

3) $y > 0$ ($q < \nu$). Then, Eq. (A.1) can be written as

$$F' = \frac{4 \sinh^2 \delta}{\sinh^2(q\sigma - \eta + \varepsilon\delta)} - \frac{\sinh^2 \kappa}{\sinh^2(\nu\sigma + \varepsilon\kappa)}. \quad (\text{A.5})$$

where $\delta > 0$ and $\kappa > 0$ are defined by

$$\begin{aligned} \frac{x}{2} &= \varepsilon \sqrt{y} \cosh \delta, & q &= \sqrt{y} \sinh \delta \\ \frac{x}{2} &= \varepsilon \frac{\sqrt{y}}{2} \cosh \kappa, & \nu &= \frac{\sqrt{y}}{2} \sinh \kappa \end{aligned}$$

($\varepsilon = \text{sign}(x)$). If $x > 0$, $\nu\sigma + \kappa$ has a zero for $\sigma = -\kappa/\nu$, with $F' \rightarrow -\infty$, so that condition (5.11) is not satisfied.

If $x < 0$, $F'(0) = (4b^2 - a^2)/a^2$ is non-negative if

$$b^2 \geq a^2/4. \quad (\text{A.6})$$

$F'(\sigma)$ can vanish only at the zeroes of

$$h \equiv f - g, \quad (\text{A.7})$$

where the functions

$$f \equiv -\sinh(\nu\sigma - \kappa), \quad g \equiv -\frac{\nu}{q} \sinh(q\sigma - \eta - \delta). \quad (\text{A.8})$$

are positive for σ negative. However $h(\sigma)$ stays positive in I_σ because the inequalities

$$(\nu - q)\sigma < 0 < \kappa - \eta - \delta \quad (\text{A.9})$$

(see below) imply

$$h' = -\nu[\cosh(\nu\sigma - \kappa) - \cosh(q\sigma - \eta - \delta)] < 0, \quad (\text{A.10})$$

so that the minimum value of h is $h(0) = (\nu/\sqrt{y})(2 + a/|b|) \geq 0$. Therefore F' stays positive in I_σ , so that (condition (5.12) being again satisfied by virtue of (A.3)) the corresponding static solutions, with parameters

$$2(a + |b|) \leq x < a - b^2/a, \quad -2|b| \leq a < -|b| \quad (\text{A.11})$$

(the first two inequalities result from (5.13) and $y > 0$, the last two from (A.6) and $x < 0$ together with (5.13)) are stable.

It remains to prove the second inequality (A.9). From the definitions of η , κ and δ we obtain

$$\begin{aligned} |b|y \sinh(\kappa - \eta - \delta) &= (\nu - q)ax - 2\nu y \\ &= 2\nu(a^2 - b^2) - (\nu + q)ax \end{aligned} \quad (\text{A.12})$$

$$\geq 2(|a| - |b|)(\nu|b| - q|a|), \quad (\text{A.13})$$

using the first inequality (A.11). The first factor is non-negative from the last inequality (A.11), while

$$\nu^2 b^2 - q^2 a^2 = (x^2/4)(b^2 - a^2) + y(a^2 - b^2/4) \geq (3b^2/4)y > 0 \quad (\text{A.14})$$

for $0 < y \leq x^2/4$.

References

- [1] T. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Phys. Math.* **K1**, 996 (1921);
O. Klein O, *Z. F. Physik***37**, 895 (1926).
- [2] P. Jordan *Ann. der Phys. Lpz.* **1**, 219 (1947);
Y.Thiry *C. R. Acad. Sci. Paris* **226**, 216 and 1881 (1948).
- [3] H. Leutwyler, *Arch. Sci.* **13**, 549 (1960).
- [4] P. Dobiasch and D. Maison D, *Gen. Rel. Grav.* **14**, 231 (1982).
- [5] A. Chodos and S. Detweiler, *Gen. Rel. Grav.* **14**, 979 (1982).
- [6] G. Gibbons and D. Wiltshire, *Ann. Phys. (N. Y.)* **167**, 201 (1986).
- [7] A. Tomimatsu, *Prog. Theor. Phys.* **74**, 630 (1985).
- [8] M. Azreg-Aïnou and G. Clément, *Gen. Rel. Grav.* **25**, 881 (1993).
- [9] M. Azreg-Aïnou, “Solutions Stationnaires en théorie de Kaluza-Klein”, *Doctoral Thesis*, Nice University (1995), gr-qc/9511021.
- [10] K.A. Bronnikov, G. Clément, C.P. Constantinidis and J.C. Fabris, *Phys. Lett. A* **243**, 121 (1998); *Grav. & Cosm.* **4**, 128 (1998).
- [11] P.G.O. Freund, *Nucl. Phys.* **B209**, 300 (1982).
- [12] D. Maison, *Gen. Rel. Grav.* **10**, 717 (1979).
- [13] G. Clément, *Gen. Rel. Grav.* **18**, 861 (1986); *Phys. Lett. A***118**, 11 (1986).
- [14] D. Rasheed, *Nucl. Phys. B* **454**, 379 (1995).
- [15] F.R. Gantmacher, “Théorie des matrices”, tome 1, Dunod, Paris (1966).
- [16] S. Deser, R. Jackiw and G. 't Hooft, *Ann. Phys. (N.Y.)* **152**, 220 (1984).
- [17] G. Clément and A. Fabbri, *Class. Quant. Grav.* **16**, 323 (1999).
- [18] M. Azreg-Aïnou and G. Clément, *Gen. Rel. Grav.* **22**, 1119 (1990).
- [19] D. Finkelstein and C.W. Misner, *Ann. Phys. (N.Y.)* **6**, 230 (1969);
J.G. Williams and R.K.P. Zia, *J. Phys. A: Math., Nucl. Gen.* **6**, 1 (1973).
- [20] G. Clément, *Annales de Phys.* **14** (6 Suppl.), 55 (1989).